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**Contract Nonr-1866(32)**

**NR - 371 - 016**

**Technical Report**

**on**

**TABLES OF THE GENERALIZED STIRLING NUMBERS  
OF THE FIRST KIND**

**by**

**William F. Pickard**

**March 1, 1963**

The research reported in this document was supported by Grant 21869 of the National Science Foundation. Publication was made possible through support extended to Cruft Laboratory, Harvard University, by the Navy Department (Office of Naval Research), the Signal Corps of the U. S. Army, and the U. S. Air Force under ONR Contract Nonr-1866(32). Reproduction in whole or in part is permitted for any purpose of the United States Government.

**Technical Report No. 404**

**Cruft Laboratory  
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Cambridge, Massachusetts**

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TABLES OF THE GENERALIZED STIRLING NUMBERS  
OF THE FIRST KIND


by

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ABSTRACT

 The generalized Stirling numbers of the first kind are defined, certain of their basic properties discussed, and tables given for the square grid  $k = 0(1)10$  and  $j = 0(1)10$  with  $l = -10(1)10$ .



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\*The work reported in this paper was supported by a grant from the National Science Foundation.

The use of the losenge diagram [1] to construct polynomial approximations, for interpolation, for the step by step integration of differential equations, for the derivation of formulae for numerical differentiation, and for other purposes, requires the evaluation of the factorial polynomials

$$(1) \quad (u-l)^{[k]} = (u-l)(u-l-1)\cdots(u-l-k+1)$$

where  $l$  and  $k$  are integers,  $k$  being non-negative and  $l$  unrestricted.

Since  $(u-l)^{[k]}$  is a polynomial of degree  $k$  it can be represented as

$$(2) \quad \begin{aligned} (u-l)^{[k]} &= {}_lS_0^k u^k + {}_lS_1^k u^{k-1} + \cdots + {}_lS_k^k \\ &= \sum_{j=0}^k {}_lS_j^k u^{k-j} \end{aligned}$$

where the coefficients  ${}_lS_j^k$  can be called the generalized Stirling numbers of the first kind in analogy with the terminology used for the numbers  ${}_0S_j^k$ .

The  ${}_lS_j^k$  are conveniently determined using the recursive relationship

$$(3) \quad {}_lS_j^{k+1} = -(l+k) {}_lS_{j-1}^k + {}_lS_j^k \quad \begin{array}{l} k = 0, 1, 2, \dots \\ j = 1, 2, 3, \dots \end{array}$$

and the obvious identities

$$(4a) \quad {}_lS_0^k = 1 \quad k = 0, 1, 2, \dots$$

$$(4b) \quad {}_lS_j^0 = 0 \quad j = 1, 2, 3, \dots$$

Equation (3) is readily derived by expanding

$$(5) \quad (u-l)^{[k+1]} = (u-l-k)(u-l)^{[k]}$$

and equating the coefficients of the several powers of  $u$ .

Only the  ${}_l S_j^k$  for  $l = 0$  have been extensively tabulated [2]; those for  $l \neq 0$  appear to have been largely neglected. In order to facilitate the construction of formulas from the lozenge diagram, the IBM 7090 at the Harvard University Computing Center was utilized to calculate the  ${}_l S_j^k$  over the square grid  $k = 0(1)10$ ,  $j = 0(1)10$  with  $l = -10(1)10$ . The results of these calculations are presented in Table I; a blank indicates that the number was zero and the word OVERFLOW that the absolute value of the number exceeded 34, 359, 738, 367.

### REFERENCES

1. K. S. Kunz, "Numerical Analysis", McGraw-Hill Book Company, Inc., New York, 1957, chapter 4.
2. A. Fletcher, J. C. P. Miller, L. Rosenhead, and L. J. Comrie, "An Index of Mathematical Tables", Second Edition, Addison-Wesley Publishing Company, Inc., Reading, 1962, section 4.9231.

$f--10$										
$k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	10								
2	1	19	90							
3	1	27	242	720						
4	1	34	431	2414	5040					
5	1	40	635	5000	19524	30240				
6	1	45	835	8175	44524	127860	151200			
7	1	49	1015	11515	77224	305956	662640	604800		
8	1	52	1162	14560	111769	537628	1580508	2592720	1814400	
9	-1	54	1266	16884	140889	761166	2655764	5753736	6999840	3628800
10	1	55	1320	18150	157773	902055	3416930	8409500	12753576	16628640
										3628800

$f--9$										
$k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	9								
2	1	17	72							
3	1	24	191	504						
4	1	30	435	1650	3024					
5	1	35	485	3325	11274	15120				
6	1	39	625	5265	24574	60216	60480			
7	1	42	742	7140	40369	133938	241128	181440		
8	1	44	826	8624	54659	214676	509004	663696	362880	
9	1	45	870	9450	63273	269325	723680	1172700	1026576	362880
10	1	45	870	9450	63273	269325	723680	1172700	1026576	362880

$f--8$										
$k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	8								
2	1	15	56							
3	1	21	146	336						
4	1	26	251	1066	1680					
5	1	30	355	2070	5944	6720				
6	1	33	445	3135	12154	24552	20160			
7	1	35	511	4025	18424	48860	69264	40320		
8	1	36	546	4536	22449	67284	118124	109584	40320	
9	1	36	546	4536	22449	67284	118124	109584	40320	
10	1	35	510	3990	17913	44835	50840	-8540	-69264	-40320

TABLE I GENERALIZED STIRLING NUMBERS OF THE FIRST KIND



		f--7										
k \ j		0	1	2	3	4	5	6	7	8	9	10
0	1											
1	1	7										
2	1	13	42									
3	1	18	107	210								
4	1	22	179	638	840							
5	1	25	245	1175	2754	2520						
6	1	27	295	1665	5104	8028	5040					
7	1	28	322	1960	6769	13132	13068	5040				
8	1	28	322	1960	6769	13132	13068	5040	5040			
9	1	27	294	1638	4809	6363	-64	-5040	-8028	-5040		
10	1	25	240	1050	1533	-3255	-12790	11016	-7900	10080		

		f--6										
k \ j		0	1	2	3	4	5	6	7	8	9	10
0	1											
1	1	6										
2	1	11	30									
3	1	15	74	120								
4	1	18	119	342	360							
5	1	20	155	580	1044	720						
6	1	21	175	735	1624	1764	720					
7	1	21	175	735	1624	1764	720					
8	1	20	154	560	889	140	-1044	-720				
9	1	18	114	252	-231	-1638	-1324	1368	1440			
10	1	15	60	-90	-987	-945	3590	-2664	-4320			

		f--5										
k \ j		0	1	2	3	4	5	6	7	8	9	10
0	1											
1	1	5										
2	1	12	20									
3	1	14	47	60								
4	1	15	71	154	120							
5	1	15	85	225	274	120						
6	1	15	85	225	274	120	120					
7	1	14	70	140	49	-154	120					
8	1	12	42	-126	-231	-252	-120	240				
9	1	9	6	-30	-231	441	188	-324	-720			
10	1	5			-150	1365	-820	-4100	576	2880		

TABLE I (CONTINUED)

{-4											
k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	4									
2	1	7	12								
3	1	9	26	24							
4	1	10	35	50	24						
5	1	10	35	50	24						
6	1	9	25	15	-26	-24					
7	1	7	7	-35	-56	28	48				
8	1	4	-14	-56	49	196	-36	-144			
9	1		-30		273	-820	4100	576	-2880		
10	1	-5	-30	150	273	-820	4100	576			

{-3											
k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	3									
2	1	5	6								
3	1	6	11	6							
4	1	6	11	6							
5	1	5	-5	-15	-6						
6	1	3	-15	-15	4	12					
7	1	-4	-14	56	49	-196	-36	144			
8	1	-9	-14	126	-231	-441	-36	324			
9	1	-15	60	90	-987	945	944	-5340	-720	4320	
10	1						3590	-2664	-2664		

{-2											
k \ j	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	2									
2	1	3	2								
3	1	3	2	-2							
4	1	2	-1	-5	4						
5	1	-3	-5	15	-4	-12					
6	1	-7	7	35	-56	-28	48				
7	1	-12	42	-231	-231	252	188	-240			
8	1	-18	114	-252	-231	1638	-1324	-1368	1440		
9	1	-25	240	-1050	1533	3255	-12790	7900	11016	-10080	
10	1										

TABLE I (CONTINUED)



$\lambda$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-2									
2	1	-5	6								
3	1	-9	26	-24							
4	1	-14	71	-154	120						
5	1	-20	155	-500	1044	-720					
6	1	-27	295	-1445	5104	-8028	5040				
7	1	-35	511	-4025	18424	-48660	69264	-40320			
8	1	-44	826	-8424	54449	-214676	509004	-643496	342840		
9	1	-54	1266	-16884	140089	-761166	2655764	-5753736	6999840	-3428000	
10	1	-65	1840	-30810	326613	-2310945	11028590	-34967140	70299936	-80627040	39916000

$\lambda$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-3									
2	1	-12	12								
3	1	-18	47	-60							
4	1	-25	119	-342	360						
5	1	-33	245	-1175	2754	-2520					
6	1	-42	445	-3135	12154	-24652	20160				
7	1	-52	742	-7140	40369	-139938	241128	-181440			
8	1	-63	1162	-14560	111769	-537628	1500508	-2592720	1814400		
9	1	-75	1734	-27342	271929	-1767087	7494416	-19978308	36354320	-19959400	
10	1		2490	-48150	600033	-5030235	28699460	-109911300	270074016	-383970040	239000000

$\lambda$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-4									
2	1	-15	20								
3	1	-22	74	-120							
4	1	-30	179	-630	840						
5	1	-39	325	-2070	5944	-6720					
6	1	-49	625	-5245	24574	-60216	60480				
7	1	-60	1015	-11515	77224	-309956	642640	-604800			
8	1	-72	1954	-22600	203009	-1155420	4028156	-7893040	4652000		
9	1	-85	3210	-41320	476049	-3460388	17893196	-54231712	101378000	-79833000	
10	1			-70090	1013313	-9790725	64720340	-280843260	832991136	-1397759040	1037836000

TABLE 1 (CONTINUED)

$k \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-5									
2	1	-11	30								
3	1	-18	107	-210							
4	1	-26	251	-1066	1680						
5	1	-35	485	-3325	11274	-15120					
6	1	-45	835	-8175	44524	-127860	151200				
7	1	-56	1330	-17360	134449	-617624	1557660	-1663200			
8	1	-68	2002	-33321	342769	-2231012	8599148	-20355120	19958400		
9	1	-81	2886	-59346	775929	-6687009	37972304	-136954044	284574960	-259459200	
10	1	-95	4020	-99750	1606773	-17550015	131590430	-668566300	2201931576	-4243508640	3632428800

$k \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-6									
2	1	-13	42								
3	1	-21	146	-336							
4	1	-30	335	-1650	3024						
5	1	-40	635	-5000	19524	-30240					
6	1	-51	1075	-11985	74524	-245004	332640				
7	1	-63	1687	-24885	218344	-1139292	3272688	-3991680			
8	1	-76	2506	-46816	41849	-3977764	18083484	-45536624	51891840		
9	1	-90	3570	-81900	1197273	-11563650	73772180	-299705400	703404576	-726485760	
10	1	-105	4920	-135450	2425773	-29522745	247226930	-1406288100	5198985576	-11277554400	10897286400

$k \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	-7									
2	1	-15	56								
3	1	-24	191	-504							
4	1	-34	431	-2414	5040						
5	1	-45	805	-7155	31594	-55440					
6	1	-57	1345	-16815	117454	-434568	665280				
7	1	-70	2086	-34300	336049	-1961470	6314664	-8648640			
8	1	-84	3066	-63504	816249	-6666156	33775244	-97053936	121080960		
9	1	-99	4326	-109494	1768809	-18909891	133767584	-603682596	1576890000	-1816214400	
10	1	-115	5910	-178710	3520713	-47210835	436325840	-2743963940	11235811536	-27046454400	29059430400

TABLE I (CONTINUED)

$\frac{j}{k}$	0	1	2	3	4	5	6	7	8	9	10
0	1	-8									
1	1	-17	72	-720							
2	1	-27	242	-3382	7920						
3	1	-38	539	-9850	48504	-95040					
4	1	-50	995	-22785	176554	-725592	1235520				
5	1	-63	1645	-45015	495544	-3197348	11393808	-17297280			
6	1	-77	2527	-83720	1182769	-10630508	59354028	-186504400	259459200		
7	1	-92	3682	-142632	2522289	-29554812	229442156	-1137868848	3270729600	-4151347200	
8	1	-108	5154	-230250	4947033	-72433725	731873960	-5038385500	22614500016	OVERFLOW	OVERFLOW
9	1	-125	6990								

$\frac{j}{k}$	0	1	2	3	4	5	6	7	8	9	10
0	1	-9									
1	1	-19	90	-990							
2	1	-30	299	-4578	11880						
3	1	-42	659	-13145	71394	-154440					
4	1	-55	1205	-30015	255424	-1153956	2162160				
5	1	-69	1975	-59440	705649	-4985316	19471500	-32432400			
6	1	-84	3010	-107800	1659889	-16275700	99236556	-343976400	518918400		
7	1	-100	4354	-181818	3492489	-44493813	375923456	-2030997852	6366517200	-8821612800	
8	1	-117	6054	-290790	6765213	-107358615	1176812090	-8797620060	OVERFLOW	OVERFLOW	OVERFLOW
9	1	-135	8160								

$\frac{j}{k}$	0	1	2	3	4	5	6	7	8	9	10
0	1	-10									
1	1	-21	110	-1320							
2	1	-33	362	-6026	17160						
3	1	-46	791	-17100	101524	-240240					
4	1	-60	1435	-38625	358024	-1763100	3603600				
5	1	-75	2335	-75985	976024	-7491484	31813200	-57657600			
6	1	-91	3535	-136080	2267769	-24083892	159168428	-598482000	980179200		
7	1	-108	5082	-227556	4717209	-64903734	592678484	-343513704	11752855200	-17643225600	
8	1	-126	7026	-361050	9040773	-154530705	1825849430	-14724404900	OVERFLOW	OVERFLOW	OVERFLOW
9	1	-145	9420								

TABLE I (CONTINUED)

